The spores, so far as observed, are all of one kind; they are ellipsoidal in form, with longitudinal crests or ridges; their dimensions are 90—96 μ in length by 65—70 μ in width.

The most characteristic point in the structure of the new cone—the fertility of both dorsal and ventral lobes of the sporophyll—is regarded as more probably due to special modification than to the retention of a primitive condition.

"On the Convergence of Infinite Series of Analytic Functions."

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(Abstract.)

Consider the differential equation

$$\frac{d^2y}{dz^2}+k^2Qy = 0,$$

where

$$Q = Q_0+\frac{Q_1}{k}+\frac{Q_2}{k^2}+\ldots,$$

$\sqrt{Q_0}$, Q_1, Q_2, . . ., are one-valued analytic functions of z, independent of k, k is a constant, and the series defining Q is convergent for all values of k, such that

$$|k| > R,$$

except at the singularities of the functions

$$Q_0, Q_1, Q_2 \ldots$$

Exclude points in the z-plane indefinitely near these singularities.

The series

$$y = e^{ik\omega}\left(\phi_0+\frac{\phi_1}{k}+\frac{\phi_2}{k^2}+\ldots \textit{ ad inf.}\right)$$

$$+e^{-ik\omega}\left(\psi_0+\frac{\psi_1}{k}+\frac{\psi_2}{k^2}+\ldots \textit{ ad inf.}\right),$$

where ω ϕ ψ are functions of z independent of k, can be formally and uniquely constructed to represent any given particular integral of the differential equation.

For all values of z not excluded the series

$$\phi_0+\frac{\phi_1}{k}+\frac{\phi_2}{k^2}+\ldots, \qquad \psi_0+\frac{\psi_1}{k}+\frac{\psi_2}{k^2}+\ldots$$

are convergent, and when k is very large,

$$e^{ik\omega}\phi_0+e^{-ik\omega}\psi_0$$

is an approximate value of the integral.

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By means of this theorem approximate expressions for large values of one or more of the arguments can be found for many functions that occur in analysis and satisfy linear differential equations of the second order. For instance, the approximate solution of Lamé's equation

$$\frac{1}{y}\frac{d^2y}{dz^2} = n(n+1)\,k^2\mathrm{sn}^2 z + \mathrm{B}$$

for large values of n is

$$\mathrm{sn}^{-\frac{1}{2}}z\left[\mathrm{C}_1(\mathrm{dn}\,z + k\,\mathrm{cn}\,z)^{n+\frac{1}{2}} + \mathrm{C}_2(\mathrm{dn}\,z + k\,\mathrm{cn}\,z)^{-n-\frac{1}{2}}\right],$$

where C_1 and C_2 are arbitrary constants.

We can often determine the condition of convergence of an infinite series of functions by replacing the nth term by its approximate value for large values of n. For instance, the series of Lamé's functions[*]

$$\sum_{n=1}^{\infty} c_n \mathrm{E}_n^p(\sin x)\,.\,\mathrm{F}_n^p(\sin t),$$

where the c's are arbitrary constants, subject to the condition that the series

$$\sum_{n=1}^{\infty} c_n z^n$$

has unit radius of convergence, converges if

$$\frac{\mathrm{dn}(\xi, k)\,.\,\mathrm{dn}(\eta, k')}{\mathrm{cn}(\xi, k)} < \frac{\mathrm{dn}(\xi_1, k)\,.\,\mathrm{dn}(\eta_1, k')}{\mathrm{cn}(\xi_1, k)},$$

where

$$x \equiv \xi + i\eta, \qquad t \equiv \xi_1 + i\eta_1.$$

The limitation of the method is noteworthy; we cannot find the condition of convergence of the expansion of a given function in an infinite series of given functions without knowing—

(i) That the expansion formally exists; and

(ii) The approximate, if not the accurate, value of the nth term in the expansion for large values of n.

In the case of an expansion in hypergeometric functions, the limitation may be removed with the help of two theorems:—

(i) If $\phi(z)$ be a solution of the linear differential equation of the nth order,

$$(a_0 + a_1 z + \ .\ .\ .\ + a_n z^n)\frac{d^n y}{dz^n} + (b_0 + b_1 z + \ .\ .\ .\ + b_{n-1}z^{n-1})\frac{d^{n-1}y}{dz^{n-1}}$$
$$+\ .\ .\ .\ + (k_0 + k_1 z)\frac{dy}{dz} + l_0 y = 0,$$

the coefficient of $\dfrac{d^r y}{dz^r}$ being a polynomial in z of order r; then

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[*] The notation is that of Byerly ('Fourier's Series and Spherical Harmonics,' p. 255 (1895)), $\mathrm{B} = p(1 + k^2)$.

$$f(z) = \int \frac{\phi(t)}{t-z} \, . \, dt,$$

taken along a suitable path, satisfies the equation, and there are in general $(n-1)$ such integrals linearly independent.

(ii) If u and v be integrals of the equations

$$\phi_1(z) \frac{d^2u}{dz^2} + \phi_2(z) \frac{du}{dz} + \alpha\phi_3(z)\, u = 0,$$

and

$$\phi_1(z) \frac{d^2v}{dz^2} + \phi_2(z) \frac{dv}{dz} + \beta\phi_3(z)\, v = 0,$$

where ϕ_1, ϕ_2, ϕ_3 are any holomorphic functions of z, and α, β are two unequal constants, the integral

$$\int_{\mathrm{L}} uv \frac{\phi_3(z)}{\phi_1(z)} \, . \, e^{\int \frac{\phi_3(z)}{\phi_1(z)} \, . \, dz} \, . \, dz$$

vanishes if L is a suitably chosen path.

We deduce the following result:—

Let $\phi(z)$ be any function of z which is regular at all points in the interior of an ellipse, C, whose foci are at the points $z = 0$ and $z = 1$. The ellipse passes through one (or more) of the singularities of $\phi(z)$. The curve is thus completely defined when $\phi(z)$ is given. Further, let p, q, γ be any constant quantities whatever, real or complex, save that neither $(p+1)$ nor γ is a negative integer. Then $\phi(z)$ can be expanded in the infinite series of hypergeometric functions.

$$\sum_{n=0}^{\infty} a_n \mathrm{F}(p+n, q-n, \gamma, z),$$

where

$$a_n = \frac{(p-q+2n) \, . \, \Pi(\gamma-q+n-1) \, . \, \Pi(p+n-1)}{\{\Pi(\gamma-1)\}^2 \, . \, \Pi(n-q) \, . \, \Pi(p-\gamma+n)}$$

$$\int^{(0+,\,1+,\,0-,\,1-)} t^{\gamma-1}(1-t)^{p+q-\gamma} . \, \mathrm{F}(p+n, q-n, \gamma, t) \, . \, \phi(t) \, . \, dt.$$

The series is convergent if z is inside C, and divergent if z is outside C; if z is on C the series is in general oscillatory and the expansion fails.